Brownian Motion

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1 Definition

A stochastic process is a set of random variables usually indexed by time. We define a stochastic process $w : \mathbb{R}^+ \to \mathbb{R}$ with the following properties as a Brownian motion:

(a) $w(0) = 0$

(b) $w(t)$ is continuous in time i.e.

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |t_1 - t_2| < \delta => |w(t_1) - w(t_2)| < \epsilon$$

(c) $w(t)$ has independent increments i.e.

$$w(t_1) - w(t_2) \perp w(t_3) - w(t_4) \text{ if } (t_1, t_2) \cap (t_3, t_4) = \{\}$$

(d) $w(t)$ is "normally" stationary i.e.

$$w(t_1) - w(t_2) \sim N(0, |t_1 - t_2|)$$

2 Properties

2.1 Distance

We want to evaluate the distance traveled by a wiener process between $[0, T]$ i.e.

$$\begin{aligned}
\int_0^T |dw(t)| &= \lim_{n\to\infty} \sum_{k=0}^{k=n-1} \left| w\left(\frac{(k+1)T}{n}\right) - w\left(\frac{kT}{n}\right) \right| \\
&= \lim_{n\to\infty} \sum_{k=0}^{k=n-1} |X_k| \ , \ X_k \sim N\left(0, \frac{T}{n}\right) \\
&= \lim_{n\to\infty} \sqrt{\frac{T}{n}} \sum_{k=0}^{k=n-1} |X_k| \ , \ X_k \sim N(0,1)
\end{aligned}$$

using law of large numbers,

$$\begin{aligned}
&= E(|X_1|) \lim_{n\to\infty} \sqrt{nT} \ , \ X_1 \sim N(0,1) \\
&= E(Y_1) \lim_{n\to\infty} \sqrt{nT} \ , \ Y_1 \sim \text{Half-Normal}(0,1) \\
&= \sqrt{\frac{2}{\pi}} \lim_{n\to\infty} \sqrt{nT} \\
&= \infty
\end{aligned}$$

2.2 Non-differentiable

We first define a random variable $m(t) = \max_{0<s<t} w(s)$. Now,

$$\begin{aligned}
P(m(t) < x) &= P(\max_{0<s<t} w(s) < x) \\
&= P(w(\tau) < x), \tau = \min\{u : \forall v \in (0,t), w(v) \leq w(u)\} \\
&= 1 - P(\tau < t), \tau = \min\{u : w(u) \geq x\} \\
&= 1 - P(\tau < t, w(t) < x) - P(\tau < t, w(t) > x), \tau = \min\{u : w(u) \geq x\} \\
&= 1 - P(\tau < t, w(t) - w(\tau) < 0) - P(\tau < t, w(t) - w(\tau) > 0), \tau = \min\{u : w(u) \geq x\} \\
&= 1 - 2P(\tau < t, w(t) - w(\tau) > 0), \tau = \min\{u : w(u) \geq x\} \\
&= 1 - 2P(w(t) > x) \\
&= 2\Phi\left(\frac{x}{\sqrt{t}}\right)) - 1
\end{aligned}$$

Suppose that $w(t)$ is differentiable i.e.

$$\begin{aligned}
\forall \delta \exists N \text{ s.t. } 0| < t_1 - t_2| < \delta => \frac{|w(t_1) - w(t_2)|}{|t_1 - t_2|} < N \\
=> |w(t_1) - w(t_2)| < N\delta
\end{aligned}$$

Next, we move to probability space,

$$\begin{aligned}
P(|w(t_1) - w(t_2)| < N\delta) &\leq P(w(t_1) - w(t_2) < N\delta) \\
&= P(m(t_1 - t_2) < N\delta) \\
&= 2\Phi\left(\frac{N\delta}{\sqrt{\delta}}\right)) - 1 \\
&= 2\Phi(N\sqrt{\delta}) - 1
\end{aligned}$$

We observe that as $RHS = \lim_{\delta \to 0} 2\Phi(N\sqrt{\delta}) - 1 = 0$ implying that $w(t)$ is nowhere differentiable.

2.3 Quadratic variation

We want to evaluate the sum of squared distance traveled by a wiener process between [0,T] i.e.

$$\begin{aligned}
\int_0^T dw(t) &= \lim_{n\to\infty} \sum_{k=0}^{k=n-1} \left[w\left(\frac{(k+1)T}{n}\right) - w\left(\frac{kT}{n}\right)\right]^2 \\
&= \lim_{n\to\infty} \sum_{k=0}^{k=n-1} X_k^2 \text{ , } X_k \sim N^2\left(0, \frac{T}{n}\right) \\
&= \lim_{n\to\infty} \frac{T}{n} \sum_{k=0}^{k=n-1} X_k^2 \text{ , } X_k \sim N(0,1)
\end{aligned}$$

using law of large numbers,

$$\begin{aligned}
&= TE(X_1^2) \text{ , } X_1 \sim N(0,1) \\
&= TE(Y_1) \text{ , } Y_1 \sim \chi_1^2 \\
&= T
\end{aligned}$$

To be consistent with this observation, we would need to use $(dw(t))^2 = dt$ in future sections.

3 Ito's calculus

Suppose that we have a C^∞ function $f : \mathbb{R} \to \mathbb{R}$. Now, we want to compute $\Delta f(w_t)$. We use Taylor series,

$$\begin{aligned}\Delta f(w_t) &= f'(w_t)\Delta w_t + f''(w_t)(\Delta w_t)^2 + f'''(w_t)(\Delta w_t)^3 + ... \\ &= f'(w_t)\Delta w_t + f''(w_t)(\Delta t)^2 + f'''(w_t)(\Delta w_t)^3 + ... \\ &= f'(w_t)\Delta w_t + f''(w_t)(\Delta t)^2 \text{ (ignoring higher order terms)}\end{aligned}$$

Now, if we have $f(t, X_t)$ where $dX_t = \mu_t dt + \sigma_t dw_t$, then we can can write,

$$\begin{aligned}df(t, X_t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{\partial^2 f}{\partial x^2}d(X_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dw_t\end{aligned}$$

3.1 Self-integral

We want to know the integral of a wiener process with itself between $[0, T]$ i.e. $\int_0^T w(t)dw(t)$. Firstly, we note that we get the below from Ito's,

$$\begin{aligned}(w(t))^2 - (w(0))^2 &= \int_0^t d(w(s))^2 \\ &= \int_0^t 2w(s)dw(s) + \int_0^t 2(dw(s))^2 \\ &= 2\int_0^t w(s)dw(s) + \int_0^t (dt)^2\end{aligned}$$

Now, we can easily see that

$$\int_0^t w(s)dw(s) = \frac{(w(t))^2}{2} - \frac{t}{2}$$

4 Martingale representation theorem

If M_t is a martingale in $(\Omega, \mathcal{F}_t, P)$ i.e.

$$E^P[M(s)|\mathcal{F}_t] = M(t)$$

then there exists an adapted process f such that

$$M(t) - M(0) = \int_0^t f(u)dw(u)$$