Numeraire

Abhinav Mehta

April 26, 2020

1 Definition

A numeraire N_t is a strictly positive adapted process on $\mathcal{F}_t$; and can be interpreted as a unit of reference. Common numeraires are:

- Money market account which is the money accumulated in a bank account i.e.

  $$N(t) = \exp\left(\int_0^t r_s ds\right)$$

  The induced equivalent martingale measure is famously called risk-neutral measure.

- Zero-coupon bond $P(t,T)$ maturing T with $P(t,T) = 1$

- A combination of zero-coupon bonds with varying maturities i.e.

  $$N_t = \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k)$$

  where $0 < T_1 < T_2 < ... < T_n$ and $0 \leq t \leq T_n$.

2 Measure change

2.1 Equivalent measure

Measures $\tilde{P}$ and P are said to be equivalent if

(a) $supp(\tilde{P}) = supp(P)$ i.e. $\tilde{P}(A) > 0 \iff P(A) > 0$, $\forall A$.

(b) There exists a stochastic process $Z = \frac{d\tilde{P}}{dP}$ s.t.

  - $Z(\omega) > 0$, $\forall \omega$
  - $E^P(Z) = \int Z(\omega) dP(\omega) = 1$.
  - $\tilde{P}(A) = \int_A Z(\omega) dP(\omega)$, $\forall A$

The below theorem helps in ensuring existence as well as finding the Radon-Nikodym derivative and the transformed process.

2.2 Girsanov theorem:

Let w_t be a brownian motion defined on $(\Omega, \mathcal{F})$ where f is an adapted process, then an equivalent process $\tilde{w}_t$ can be defined as

$$\tilde{w}(t) = w(t) + \int_0^t f(s) ds$$

where the Radon-Nikodym derivative is defined as

$$Z(t) = \frac{d\tilde{P}}{dP} = \exp\left\{\int_0^t f(s) dw(s) - \frac{1}{2}\int_0^t f(s)^2 ds\right\}$$

2.3 Deflated process

Suppose that we have an asset S_t. We define a contingent claim V_t on S which pays $V(T)$ at maturity T. Assuming no arbitrage and completeness, the deflated process of V_t would be a martingale under some numeraire N_t and measure $\tilde{P}$ i.e.

$$\frac{V(t)}{N(t)} = E^{\tilde{P}}\left(\frac{V(T)}{N(T)} \mid \mathcal{F}_t\right)$$

3 Black-Scholes-Merton option formula

We assume that in the physical measure, the below process is taken by a non-dividend paying stock,

$$d\tilde{S}_t = \mu \tilde{S}_t dt + \sigma \tilde{S}_t d\tilde{w}_t$$

where μ, σ and r are assumed to be constant over time.

Assuming $\theta = \frac{\mu - r}{\sigma}$. Under risk neutral measure,

$$\begin{aligned} dw(t) &= d\tilde{w}(t) + \theta dt \\ Z(t) &= \exp\left\{\int_0^t \theta d\tilde{w}(s) - \frac{1}{2}\int_0^t \theta^2 ds\right\} \end{aligned}$$

We can rewrite the stock process as

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t) d\tilde{w}(t) \\ &= \mu S(t)dt + \sigma S(t)\left(dw(t) - \theta dt\right) \\ &= \mu S(t)dt + \sigma S(t)\left(dw(t) - \frac{\mu - r}{\sigma}dt\right) \\ &= rS(t)dt + \sigma S(t)dw(t) \end{aligned}$$

We now work in risk neutral measure. To solve the above, we start with

$$\begin{aligned} d(ln(S_t)) &= \frac{\partial ln(S_t)}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 ln(S_t)}{\partial {S_t}^2}(dS_t)^2 \\ &= \frac{1}{S_t}(rS_tdt + \sigma S_t dw_t) + \frac{-1}{2S_t^2}(rS_tdt + \sigma S_t dw_t)(rS_tdt + \sigma S_t dw_t) \\ &= rdt + \sigma dw_t - \frac{\sigma^2}{2}dt + \mathcal{O}(dt^2) + \mathcal{O}(dt.dw) \\ &= \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dw_t \end{aligned}$$

Finally, we have,

$$\begin{aligned} S_t &= S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\int_0^t dw_t\right\} \\ &= S_0 \exp\left\{\sigma w_t + \left(r - \frac{\sigma^2}{2}\right)t\right\} \\ &= S_0 \exp\left\{\sigma\sqrt{t}X_t + \left(r - \frac{\sigma^2}{2}\right)t\right\}, X_t \sim N(0,1) \end{aligned}$$

The price of a call option can be written as

$$
\begin{aligned}
Call(t, S_t) &= E\left[e^{-rt}|S_t - K|^+\right] \\
&= \frac{e^{-rt}}{\sqrt{2\pi}} \int_{\infty}^{\infty} \left| S_0 \exp\left\{\sigma x \sqrt{t} + \left(\mu - \frac{\sigma^2}{2}\right) t\right\} - K \right|^+ e^{\frac{-x^2}{2}} dx
\end{aligned}
$$

which solves to the famous Black-Scholes-Merton formula.

4 Generalizations

The Black-Scholes-Merton model makes a lot of simplifying assumptions which are not observed in reality. In response to that, we would like to price under arbitrary numeraire where N_t follows the below process in the physical measure

$$d\tilde{N}(t) = \mu(t)\tilde{N}(t)dt + v(t)\tilde{N}(t)d\tilde{w}(t)$$

Under risk neutral measure, this can be transformed as

$$dN(t) = r(t)N(t)dt + v(t)N(t)dw(t)$$

Using the method employed in the earlier section and defining $R(t) = \exp\{\int_0^t -r(s)ds\}$, we get

$$
\begin{aligned}
N(t) &= N(0) \exp\left\{\int_0^t \left(r(s) - \frac{v(s)^2}{2}\right) ds + \int_0^t v(s) dw_s\right\} \\
\implies \frac{R(t)N(t)}{N(0)R(0)} &= \exp\left\{\int_0^t \left(-\frac{v(s)^2}{2}\right) ds + \int_0^t v(s) dw_s\right\}
\end{aligned}
$$

The above can be treated as Radon-Nikodym derivative. Using Girsanov theorem, the new wiener process is

$$dw_t^N = dw + v_t dt$$

and the corresponding probability measure is

$$P_t^N(A) = \int_A \frac{N(t)R(t)}{N(0)R(0)} dP_t \text{ , where } A \in \mathcal{F}_t$$

4.1 Correlation

Suppose that we have two correlated processes in the physical measure

$$
\begin{aligned}
d\tilde{S}_1(t) &= \mu_1(t)\tilde{S}_1(t)dt + \sigma_1(t)\tilde{S}_1(t)d\tilde{w}_1(t) \\
d\tilde{S}_2(t) &= \mu_2(t)\tilde{S}_2(t)dt + \sigma_2(t)\tilde{S}_2(t)\left[\rho d\tilde{w}_1(t) + \sqrt{1-\rho^2} d\tilde{w}_2(t)\right]
\end{aligned}
$$

It would be straightforward transformation if they were denominated under the same money market account. Suppose that the corresponding money market accounts are $N(t) = \exp\{-\int_0^t n(s)ds\}$ and $M(t) = \exp\{-\int_0^t m(s)ds\}$ respectively. Then the exchange rate would be $Q(t) = Q(0)\exp(\int_0^t (m(s) - n(s))ds)$. We want to move them under a common money market numeraire N. For S_1, we can refer to the earlier sections to write the below

$$dS_1^N(t) = n(t)S_1^N(t)dt + \sigma_1(t)S_1^N(t)dw_1^N(t)$$

For S_2, we first use the martingale representation theorem which states that

$$\forall t \exists \Gamma(t) \text{ s.t. } d\left(Q(t)N(t)S_2^N(t)\right) = \Gamma(t)dw^N(t)$$

We define $\sigma_2(t) = \frac{\Gamma(t)}{Q(t)N(t)S_2^N(t)}$, then

$$d\left(Q(t)N(t)S_2^N(t)\right) = \sigma_2(t)Q(t)N(t)S_2^N(t)dw^N(t)$$