Poisson Process

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1 Definition

Poisson process assumes a counting process $\{N_t;\, t \in (0, \infty);\, \text{support} = n \in \mathbb{N}\}$ to be a random variables with below properties:

(a) $N(0) = 0$

(b) $N(t) - N(s) \perp N(y) - N(x)$ if $(s, t) \cap (x, y) = \{\}$

(c) $N(t) - N(s) \sim \text{Poisson}(\lambda(t - s))$

2 Properties

2.1 New arrival time

Let us define a stochastic process X which denotes the next arrival time in a Poisson process i.e. $X_t = \min\{\tau : N(t + \tau) - N(t) > 0\}$

$$\begin{aligned} P(X_t < x) &= P(N(t + x) - N(t) > 0) \\ &= 1 - P(N(t + x) - N(t) = 0) \\ &= 1 - \frac{(\lambda x)^0}{0!} e^{-\lambda x} \\ &= 1 - e^{-\lambda x} \end{aligned}$$

Finally, we observe that new arrival time has an Exponential distribution i.e. $X_t \sim \text{Exp}(\lambda)$.

2.2 K^{th} arrival time

Let us define a stochastic process Y which denotes the k^{th} arrival time in a Poisson process i.e. $Y_{t,k} = \min\{\tau : N(t + \tau) - N(t) = k\}$

$$
\begin{aligned}
P(Y_{t,k} < x) &= P(N(t+x) - N(t) \geq k) \\
&= 1 - P(N(t+x) - N(t) < k) \\
&= 1 - e^{-\lambda x} \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} \\
&= e^{-\lambda x} \left(e^{\lambda x} - \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} \right) \\
&= e^{-\lambda x} \left(\sum_{j=0}^{\infty} \frac{(\lambda x)^j}{j!} - \sum_{i=0}^{k-1} \frac{(\lambda x)^i}{i!} \right) \\
&= e^{-\lambda x} \left(\sum_{j=k}^{\infty} \frac{(\lambda x)^j}{j!} \right)
\end{aligned}
$$

We observe that $Y_{t,k} \sim \text{Gamma}(k, \lambda)$. We conclude that k^{th} arrival time in a Poisson process follows Gamma distribution.

2.3 Compensated poisson process

We introduce $M(t) = N(t) - \lambda t$ as Compensated Poisson process. We observe that

$$E(M_t) = E(N_t - N_0) - \lambda t = 0$$

3 Equivalence

We begin by assuming assume independent, identical and exponentially distributed random variables i.e. $\{X_i \sim Exp(\lambda);\ i \in \mathbb{N};\ \text{support} = t \in (0, \infty)\}$. We prove that it is the same as the standard Poisson counting process we defined earlier.

(1a) $P(N(0) = 0) = 1$ is trivially true as

$$P(N(0) = k) = \sum_{i=1}^{k} P(X_i \leq 0) = 0,\ \forall k \in \mathbb{N}$$

(1b) Without loss of generality, we take $\{i, j, r, m\} \subset \mathbb{N}$ such that $\{i, ..., i + r - 1\} \cap \{j, ..., j + m - 1\} = \{\}$. And since X_i's are assumed to be independent, it implies that,

$$
\begin{aligned}
&P((N(t) - N(s)) > r \cap (N(y) - N(x)) > m) \\
&= P((X_i + X_{i+1} + ... + X_{i+r-1}) < (t - s) \cap (X_i + X_{i+1} + ... + X_{i+r}) < (t - s)) \\
&= P\left((X_i + X_{i+1} + ... + X_{i+r-1}) < (t - s)\right) P\left(X_j + X_{j+1} + ... + X_{i+m-1} < (y - x)\right) \\
&= P(N(t) - N(s) > r) P(N(y) - N(x) > m)
\end{aligned}
$$

We observe that $P((N(t) - N(s)) > r \cap (N(y) - N(x)) > m) = P(N(t) - N(s) > r)P(N(y) - N(x) > m)$ implying that N_t's increments are independent.

(1c) Without loss of generality, we take $\{i, r\} \subset \mathbb{N}$. Now,

$$\begin{aligned}
&P(N(t) - N(s) \leq r) \\
&= 1 - P(N(t) - N(s) > r) \\
&= 1 - P\left((X_i + X_{i+1} + ... + X_{i+r}) < (t - s)\right) \\
&= 1 - e^{-\lambda(t-s)} \left(\sum_{j=r+1}^{\infty} \frac{(\lambda(t-s))^j}{j!} \right) \\
&= e^{-\lambda(t-s)} \left(e^{\lambda(t-s)} - \sum_{j=r+1}^{\infty} \frac{(\lambda(t-s))^j}{j!} \right) \\
&= e^{-\lambda(t-s)} \left(\sum_{i=0}^{\infty} \frac{(\lambda(t-s))^i}{i!} - \sum_{j=r+1}^{\infty} \frac{(\lambda(t-s))^j}{j!} \right) \\
&= e^{-\lambda(t-s)} \left(\sum_{i=0}^{r} \frac{(\lambda(t-s))^i}{i!} \right)
\end{aligned}$$

We observe that $N(t) - N(s) \sim \text{Poisson}(\lambda(t-s))$.